Int. J. Solids Structures, 1970, Vol. 6, pp. 1413 to 1428. Pergamon Press. Printed in Great Britain

A CURVED FINITE ELEMENT FOR THIN ELASTIC SHELLS†

G. DUPUIS‡ and J.-J. GOËL§

Ecole Polytechnique Fédérale-Lausanne, Department of Mathematics,
33 av. de Cour, Lausanne, Switzerland

Abstract—This paper is concerned with a curved triangular finite shell element, which represents the rigid-body motions exactly and assures convergence in energy. The stiffness matrix is derived in a general way that is valid for all mathematical models which accept Kirchhoff's assumption. A numerical example is presented to indicate the quality of results that can be obtained with 9 or 18° of freedom at each meshpoint and basic functions of classes C^1 or C^2.

1. INTRODUCTION

THE APPLICATION of the finite element method to shell problems has been the object of many papers. Leaving aside cases which are essentially one-dimensional by symmetry considerations, problems may be classed in three groups.

1. The most widely used method replaces the shell by a polyhedron and treats each face as a plate element (see [1–5]). Approaches of this kind differ from each other by the choice of shape functions and by the connections imposed between the elements. Note that these connections concern the nodal displacements and do not automatically ensure continuity of displacements along the sides of the elements. Some comparisons with exact solutions show that, in many cases, approximations of this kind are sufficient for engineering purposes. It should be noted, however, that this approach is without any mathematical support. It is not justifiable as an application of Ritz's method, because the functions used do not have the required continuity. Moreover, the relation to the general theories of thin elastic shells is tenuous, because these theories concern shells with smooth middle surfaces.

2. Another method treats the shell problem as a three-dimensional one, and uses curved finite elements which are called isoparametric (see [6–8]). This procedure, which is esentially used in arch dam problems, is primarily reserved for the relatively thick shells. In the same way, Ahhad [9, 10] proposed a method, in which the thickness of the shell plays a privileged role with respect to the other dimensions of the elements. This method, however, does not seem to be satisfactory when the shell becomes thin.

3. Some curved finite elements based on two-dimensional shell theory have been used (see [11–15]). They do not however, assure the continuity of displacements, or displacements derivatives, along the sides of the elements and do not represent the rigid-body motions exactly. Some numerical investigations concerning beam problems show that the last condition is essential for good numerical results. This remark has been confirmed

† This study was initiated while the authors were at the Ecole Polytechnique Fédérale-Lausanne (EPFL, Switzerland) and was supported by the National Foundation of Scientific Research (of Switzerland). The present manuscript was prepared at Brown University in connection with a program of research sponsored by the Office of Naval Research, under Contract N00014-68-A-0191-0007.

‡ Now at: Brown University, Division of Engineering, Providence, R.I.02912, U.S.A.

§ Now at: the University of Ottawa, Faculty of Pure and Applied Science, Ottawa, Ontario, Canada.

theoretically in [16]. Contrary to what has occasionally been stated in the literature, the condition that rigid-body motions should be properly represented is essential, not for convergence in energy [17], but for acceptable rate of convergence. If this condition is fulfilled, it can show that the stresses and reactions computed from the approximate displacements assure the equilibrium of the shell, and this is, of course, of great practical importance.

In this paper, we construct a triangular shell element that guarantees convergence in energy and satisfies the condition of rigid-body motions, according to the following statements:

1. The unknown functions are the Cartesian components of the displacement.
2. The middle surface of the shell in both the undeformed and deformed states are defined, in Cartesian coordinates, as linear combinations of the same set of basic functions.
3. The strain energy vanishes exactly for all rigid-body motions of the middle surface.
4. The basic functions satisfy the conditions for convergence in energy.

In the following we shall make use of three types of basic functions; with one of them, the continuity conditions for the stress field are automatically satisfied.

Various mathematical models that are based on Kirchhoff's assumption differ in the expression of the extension and bending strains and in the constitutive equation. One of these models is therefore characterized by the matrices **A** and **B** of the strain–displacement equations, the matrix **K** of the stress–strain relation and the boundary conditions. In fact, in view of the variational formulation, a model is completely defined by the three matrices **A**, **B** and **K**. The kinematical conditions are the same for all models of this class and the statical conditions are the natural boundary conditions of the variational problem.

We shall consider here the model proposed by Koiter (see [18, 19]), which is briefly surveyed in the second section. In Section 3, we obtain the expression of the strain energy in Cartesian coordinates, from which we form the matrices **A**, **B** and **K**. Section 4 deals with the discretisation of the boundary value problem while Section 5 shows how to form the stiffness matrix of the element. An illustrative numerical example is given in the last section.

2. BASIC EQUATIONS

We give below an abstract of the basic equations of the Koiter's theory of thin shells (see [18, 19]), using the usual notations of tensor calculus† (see, for example, [20]).

Let Σ be the middle surface of the shell, defined by the equation $\mathbf{r} = \mathbf{r}(\theta^1, \theta^2)$; $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$ the base vectors; $\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2/|\mathbf{a}_1 \times \mathbf{a}_2|$ the normal to Σ; $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ and $b_{\alpha\beta} = \frac{1}{2}(\mathbf{a}_{\alpha,\beta} + \mathbf{a}_{\beta,\alpha}) \cdot \mathbf{a}_3$ the two fundamental quadratic forms on Σ. The shell considered is the volume defined by the equation $\mathbf{R}(\theta^1, \theta^2, \theta^3) = \mathbf{r}(\theta^1, \theta^2) + \theta^3 \mathbf{a}_3$, where $(\theta^1, \theta^2) \in D$, $-h/2 \leq \theta^3 \leq h/2$; D is a domain of the plane (θ^1, θ^2) and h is the thickness of the shell.

The displacement of the middle surface Σ is defined by the vector field

$$\mathbf{v} = \mathbf{v}_\alpha \mathbf{a}^\alpha + w\mathbf{a}_3, \tag{1}$$

where $\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta$ are the contravariant base vectors and $((a^{\alpha\beta})) = ((a_{\alpha\beta}))^{-1}$ is the contravariant tensor metric. It is convenient for the following to introduce the antisymmetric

† In this paper, Greek indices have the range 1, 2, a single stroke stands for covariant differentiation with respect to the surface metric and a comma denotes partial differentiation with respect to θ^α.

tensor

$$\omega_{\alpha\beta} = \tfrac{1}{2}(v_{\beta|\alpha} - v_{\alpha|\beta}) \tag{2}$$

which expresses the rotation of the middle surface around the normal. After deformation, the normal $\mathbf{a}_3$ becomes the vector $\bar{\mathbf{a}}_3 = \mathbf{a}_3 + u_\alpha \mathbf{a}^\alpha$; Kirchhoff's hypothesis yields the relation

$$u_\alpha = -(w_{,\alpha} + b^\beta_\alpha v_\beta) \tag{3}$$

The deformation of the shell is characterized by the two symmetric tensors

$$\begin{aligned} \varepsilon_{\alpha\beta} &= \tfrac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha} - 2b_{\alpha\beta}w), \\ \rho_{\alpha\beta} &= \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha} - b^\gamma_\alpha \omega_{\beta\gamma} - b^\gamma_\beta \omega_{\alpha\gamma}) \end{aligned} \tag{4}$$

which respectively represent the extension of the middle surface and the variation of its curvature. The strain parameters have a very simple intrinsic significance. Let us calculate the two fundamental forms $\bar{a}_{\alpha\beta}$ and $\bar{b}_{\alpha\beta}$ on the deformed surface $\bar{\Sigma}$; keeping only the linear terms in the displacement, we get

$$\begin{aligned} \varepsilon_{\alpha\beta} &= \tfrac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}), \\ \rho_{\alpha\beta} &= -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) + \tfrac{1}{2}(b^\gamma_\alpha \varepsilon_{\beta\gamma} + b^\gamma_\beta \varepsilon_{\alpha\gamma}) \end{aligned} \tag{5}$$

These relations show that, by a fundamental theorem of differential geometry of surfaces, strains vanish identically for all linearized rigid-body motions of the middle surface.

In the considered model, the strain energy density has the form

$$W = \frac{1}{2} B^{\alpha\beta\gamma\delta}\left(h\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + \frac{h^3}{12}\rho_{\alpha\beta}\rho_{\gamma\delta}\right),$$

where

$$B^{\alpha\beta\gamma\delta} = G(a^{\alpha\delta}a^{\beta\gamma} + a^{\alpha\gamma}a^{\beta\delta} + 2\nu/(1-\nu)a^{\alpha\beta}a^{\gamma\delta}) \tag{6}$$

with $G = E/2(1+\nu)$, E being the elastic modulus and ν the Poisson ratio. It is shown in [18] that, within the three-dimensional theory of elasticity, the expression (6) is a consistent approximation with the hypothesis of the conservation of normals. The strain energy of the shell is

$$U_1 = \int\int_\Sigma W \, \mathrm{d}\sigma \tag{7}$$

The state of stress of the shell is characterized by the symmetric tensors $n^{\alpha\beta}$ and $m^{\alpha\beta}$, defined by

$$n^{\alpha\beta} = \partial W/\partial \varepsilon_{\alpha\beta}, \qquad m^{\alpha\beta} = \partial W/\partial \rho_{\alpha\beta} \tag{8}$$

$n^{\alpha\beta}$ and $m^{\alpha\beta}$ are the two-dimensional membrane and bending stresses. From relations (6) and (8), we find the constitutive equation

$$n^{\alpha\beta} = hB^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}, \qquad m^{\alpha\beta} = \frac{h^3}{12}B^{\alpha\beta\gamma\delta}\rho_{\gamma\delta} \tag{9}$$

The external loads acting on the shell are a surface load of density $\mathbf{p} = p^\alpha \mathbf{a}_\alpha + p\mathbf{a}_3$, applied to the middle surface Σ; a line load of density $\mathbf{q} = q^\alpha \mathbf{a}_\alpha + q\mathbf{a}_3$ and a couple of density $\mathbf{m} = m_\alpha \mathbf{a}^\alpha$, both applied to the boundary Γ of Σ. The line force and couple are given on Γ_1;

they are reactive forces on $\Gamma_2 (\Gamma = \Gamma_1 \cup \Gamma_2)$. The potential of external loads is given by

$$U_2 = \int\int_{\Sigma} (p^\alpha v_\alpha + pw)\, d\sigma + \int_{\Gamma} (q^\alpha v_\alpha + qw + e^{\alpha\beta} u_\alpha m_\beta)\, ds \tag{10}$$

where $e^{\alpha\beta}$ is the antisymmetric tensor $e^{12} = -e^{21} = 1/\sqrt{a}$, $e^{11} = e^{22} = 0$.

The relations (7) and (10) define the potential energy of the shell $U = U_1 - U_2$, that is the quadratic functional of the displacements v_α and w

$$U[v_\alpha, w] = \int\int_{\Sigma} \left[\frac{1}{2} B^{\alpha\beta\gamma\delta}\left(h\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + \frac{h^3}{12}\rho_{\alpha\beta}\rho_{\gamma\delta}\right) - (p^\alpha v_\alpha + pw)\right] d\sigma$$

$$- \int_{\Gamma} (q^\alpha v_\alpha + qw + e^{\alpha\beta} u_\alpha m_\beta)\, ds \tag{11}$$

in this expression the components of the rotation u_α are defined by (3) and the strains $\varepsilon_{\alpha\beta}$ and $\rho_{\alpha\beta}$ by equations (4).

The position of equilibrium of the shell is defined by the condition

$$\delta U = 0 \tag{12a}$$

and by the geometrical boundary conditions on Γ_2. In the three simplest cases, these boundary conditions are

$v_\alpha = 0, \quad w = 0, \quad u_n = 0$	along a clamped edge, where u_n is the normal rotation;	
$v_\alpha = 0, \quad w = 0$	along a supported edge;	(12b)
no kinematic condition	along a free edge.	

From relations (12) there follow the equilibrium equation in D, the natural boundary conditions on ∂D_1 [image of Γ_1 in the plan (θ^1, θ^2)] and the forces of reaction on ∂D_2. The equilibrium equations so obtained coincide with the exact two-dimensional equilibrium equations given by Green and Zerna [20], if the tensor $m^{\alpha\beta}$ is supposed to be symmetric. It follows that the stresses solution of our boundary value problem ensure the equilibrium of all parts of the shell defined by $(\theta^1, \theta^2) \in B \subset D$, $-h/2 \leq \theta^3 \leq h/2$.

3. STRAIN ENERGY IN CARTESIAN COORDINATES

Let (x_1, x_2, x_3) be a system of Cartesian coordinates, we define the middle surface Σ by the equation $x_3 = x_3(x_1, x_2)$ or $\mathbf{r} = \mathbf{r}(x_1, x_2)$, with $\mathbf{r}^T = [x_1, x_2, x_3(x_1, x_2)]$. In order to simplify the writing, we shall use in the following the notations $z = x_3$, $z_\alpha = x_{3,\alpha}$, $z_{\alpha\beta} = x_{3,\alpha\beta}$. The base vectors on Σ can be written in that case

$$\mathbf{a}_1^T = (1, 0, z_1), \qquad \mathbf{a}_2^T = (0, 1, z_2), \qquad \mathbf{a}_3^T = \frac{1}{\sqrt{a}}(-z_1, -z_2, 1) \tag{13}$$

with $a = 1 + z_1^2 + z_2^2$. One deduces from them the two fundamental forms on Σ

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta = \delta_{\alpha\beta} + z_\alpha z_\beta, \; b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}_3 = \frac{z_{\alpha\beta}}{\sqrt{a}} \tag{14}$$

where $\delta_{\alpha\beta}$ is Kronecker's symbol.

Let u_1, u_2, u_3 be the Cartesian components of the displacement,† the deformed surface $\bar{\Sigma}$ is defined by the equation $\bar{\mathbf{r}} = \bar{\mathbf{r}}(x_1, x_2)$, where $\bar{\mathbf{r}}^T = [x_1 + u_1(x_1, x_2),\ x_2 + u_2(x_1, x_2),\ z + u_3(x_1, x_2)]$. On $\bar{\Sigma}$, the base vectors take the form

$$\begin{aligned} \bar{\mathbf{a}}_1^T &= (1 + u_{1,1}, u_{2,1}, z_1 + u_{3,1}), \\ \bar{\mathbf{a}}_2^T &= (u_{1,2}, 1 + u_{2,2}, z_2 + u_{3,2}); \end{aligned} \tag{15}$$

from this, we can find the fundamental forms $\bar{a}_{\alpha\beta}$ and $\bar{b}_{\alpha\beta}$. On keeping only the linear terms in u_i and their derivatives, we get

$$\begin{aligned} \bar{a}_{\alpha\beta} - a_{\alpha\beta} &= u_{\alpha,\beta} + u_{\beta,\alpha} + z_\alpha u_{3,\beta} + z_\beta u_{3,\alpha}; \\ \bar{b}_{\alpha\beta} - b_{\alpha\beta} &= \frac{1}{\sqrt{a}}[z_{\alpha\beta}(z_\gamma z_\lambda u_{\gamma,\lambda} - z_\gamma u_{3,\gamma})/a - z_\gamma u_{\gamma,\alpha\beta} + u_{3,\alpha\beta}]. \end{aligned} \tag{16}$$

Formulas (5) and (16) define the strains as functions of the displacement. Let us introduce the following notations:

$$\boldsymbol{\varepsilon}^T = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}), \qquad \boldsymbol{\rho}^T = (\rho_{11}, \rho_{22}, \rho_{12}),$$
$$\mathbf{n}^T = (n^{11}, n^{22}, n^{12}), \qquad \mathbf{m}^T = (m^{11}, m^{22}, m^{12});$$

and let $\boldsymbol{\partial}_x^n$ be the symbolic vector of dimension $n (n = 3 \text{ or } 6)$, defined as

$$\boldsymbol{\partial}_x^{6T} = (1, \partial x_1, \partial x_2, \partial x_1 \partial x_1, \partial x_2 \partial x_2, \partial x_1 \partial x_2);$$

$\boldsymbol{\partial}_x^3$ will be the three first components of this vector. If no confusion is possible, we shall write $\boldsymbol{\partial}^n$ rather than $\boldsymbol{\partial}_x^n$; in the same way, we shall omit the subscript n if it is not necessary to the understanding. The notation ∂_k will be the kth component of this vector. With these notations, the strain parameters can be set under the form

$$\begin{aligned} \boldsymbol{\varepsilon} &= (\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)\begin{pmatrix} \boldsymbol{\partial} u_1 \\ \boldsymbol{\partial} u_2 \\ \boldsymbol{\partial} u_3 \end{pmatrix} = \mathbf{A}\begin{pmatrix} \boldsymbol{\partial} u_1 \\ \boldsymbol{\partial} u_2 \\ \boldsymbol{\partial} u_3 \end{pmatrix}, \\ \boldsymbol{\rho} &= (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)\begin{pmatrix} \boldsymbol{\partial} u_1 \\ \boldsymbol{\partial} u_2 \\ \boldsymbol{\partial} u_3 \end{pmatrix} = \mathbf{B}\begin{pmatrix} \boldsymbol{\partial} u_1 \\ \boldsymbol{\partial} u_2 \\ \boldsymbol{\partial} u_3 \end{pmatrix} \end{aligned} \tag{17}$$

where $\boldsymbol{\partial}$ is written for $\boldsymbol{\partial}_x^6$ and the matrices **A** and **B**, of dimension 3×18, are the functions of $z, z_\alpha, z_{\alpha\beta}$ given in Table 1.

Let us introduce in (6) the contravariant components of the metric deduced from (14), we get for the constitutive equation (9)

$$\mathbf{n} = C_m \mathbf{K} \boldsymbol{\varepsilon}, \qquad \mathbf{m} = C_f \mathbf{K} \boldsymbol{\rho} \tag{18}$$

where **K** is the 3×3 matrix given in Table 2 and $C_m = Eh/(1 - \nu^2)$, $C_f = Eh^3/12(1 - \nu^2)$. The physical components of stresses, that is to say, those relative to unit base vectors, are

$$n_{(\alpha\beta)} = n^{\alpha\beta} c_{(\alpha\beta)}, \qquad m_{(\alpha\beta)} = m^{\alpha\beta} c_{(\alpha\beta)} \tag{19}$$

† Care will be taken to not confuse the Cartesian components of the displacement and the rotations defined by equation (3), which will not appear in the rest of this paper.

TABLE 1

$\mathbf{A}_1 =$

	1	0			
	0	0			
	0	1/2			

$\mathbf{A}_2 =$

	0	0			
	0	1			
	1/2	0			

$\mathbf{A}_3 =$

	z_1	0			
	0	z_2			
	$z_2/2$	$z_1/2$			

$\mathbf{B}_1 = \frac{1}{a^{3/2}}$

	$-z_{11}z_1^2 + \alpha_1$	$-z_{11}z_1z_2 + \alpha_2/2$	az_1		
	$-z_{22}z_1^2$	$-z_{22}z_1z_2 + \alpha_4/2$		az_1	
	$-z_{12}z_1^2 + \alpha_4/2$	$-z_{12}z_1z_2 + (\alpha_1+\alpha_3)/4$			az_1

$\mathbf{B}_2 = \frac{1}{a^{3/2}}$

	$-z_{11}z_1z_2 + \alpha_2/2$	$-z_{11}z_2^2$	az_2		
	$-z_{22}z_1z_2 + \alpha_4/2$	$-z_{22}z_2^2 + \alpha_3$		az_2	
	$-z_{12}z_1z_2 + (\alpha_1+\alpha_3)/4$	$-z_{12}z_1^2 + \alpha_2/2$			az_2

TABLE 1—(continued)

$\mathbf{B}_3 = \frac{1}{a^{3/2}}$

	$z_{11}z_1 + \alpha_1 z_1 + \alpha_2 z_2/2$	$z_{11}z_2 + \alpha_2 z_1/2$	-1		
	$z_{22}z_1 + \alpha_4 z_2/2$	$z_{22}z_2 + \alpha_3 z_2 + \alpha_4 z_1/2$		-1	
	$z_{12}z_1 + \alpha_4 z_1/2 + (\alpha_1+\alpha_3)z_2/4$	$z_{12}z_2 + \alpha_2 z_2/2 + (\alpha_1+\alpha_3)z_1/4$			-1

$\alpha_1 = (1+z_2^2)z_{11} - z_1 z_2 z_{12}$; $\alpha_2 = (1+z_1^2)z_{12} - z_1 z_2 z_{11}$;
$\alpha_3 = (1+z_1^2)z_{22} + z_1 z_2 z_{12}$; $\alpha_4 = (1+z_2^2)z_{12} - z_1 z_2 z_{22}$.

where $c_{(11)} = \sqrt{[a(1+z_1^2)/(1+z_2^2)]}$, $c_{(22)} = \sqrt{[a(1+z_2^2)/(1+z_1^2)]}$, $c_{(12)} = \sqrt{a}$. Both formulas (19) must be understood without sum on the indices α and β.

The strain energy of the shell can now be written as

$$U_1 = \frac{1}{2}\int\int_D \sum_{i,j=1}^{3} \partial^T u_i \mathbf{R}_{ij} \partial u_j \sqrt{a}\, dx_1\, dx_2 \tag{20}$$

where

$$\mathbf{R}_{ij} = C_m \mathbf{A}_i^T \mathbf{K} \mathbf{A}_j + C_f \mathbf{B}_i^T \mathbf{K} \mathbf{B}_j \tag{21}$$

is a 6×6 matrix only depending on the geometry of the surface (the functions z, z_α, $z_{\alpha\beta}$) and on the elastic coefficients C_m and C_f; D is the projection of the shell in the plane (x_1, x_2). In the same way, we could find the expression of the potential of the external loads, in Cartesian coordinates.

4. DISCRETISATION OF THE PROBLEM

Let us divide the domain D into triangular elements, approximating the curved parts of ∂D by straight segments; we denote by $\bar{D}$ the polygonal domain so formed and by $\partial\bar{D}$ its boundary. Let N be the number of nodes of the mesh; $\bar{D}_i$ the domain formed by the triangles admitting P_i for vertex (see Fig. 1) and $\psi_i(x_1, x_2)n$ functions associated with the node P_i,

TABLE 2

$\mathbf{K} = \frac{1}{a^2}$

$(1+z_2^2)^2$	$(1-\nu)z_1^2 z_2^2 + \nu(1+z_1^2)(1+z_2^2)$	$-2(1+z_2^2)z_1 z_2$
$(1-\nu)z_1^2 z_2^2 + \nu(1+z_1^2)(1+z_2^2)$	$(1+z_1^2)^2$	$-2(1+z_1^2)z_1 z_2$
$-2(1+z_2^2)z_1 z_2$	$-2(1+z_1^2)z_1 z_2$	$2(1-\nu)(1+z_1^2)(1+z_2^2) + 2(1+\nu)z_1^2 z_2^2$

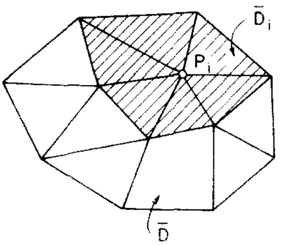

FIG. 1.

having the following properties:

1. They vanish outside of the domain $\bar{D}_i$.

2. They verify the conditions $\partial_k \psi_{il}(P_j) = \delta_{ij}\delta_{kl}$, where ψ_{il} is the lth component of the vector $\boldsymbol{\psi}_i$, ∂_k is the kth component of $\boldsymbol{\partial}_x^n$, P_j is a node of the domain $\bar{D}$ and δ_{ij} the Kronecker symbol.

3. The functions $\boldsymbol{\psi}_i$ are of class C^1 with piecewise continuous partial derivatives of second order and square integrable. They satisfy the conditions of convergence in energy, relative to the variational problem of second order. These conditions are given in [17]; we recall them for the clearness of the following.

The basic functions $\boldsymbol{\psi}_i$ assure convergence in energy of variational problems of second order, if and only if, for all polynomials of second order $Q(x_1, x_2)$, one has the relation

$$\sum_{i=1}^{N} \boldsymbol{\partial}^T Q(P_i) \, . \, \boldsymbol{\psi}_i(x_1, x_2) = Q(x_1, x_2) \tag{22}$$

In particular, this relation is of course verified for all polynomial of the first order in x_1, x_2.

In the following, we consider a shell whose middle surface is of the form

$$z(x_1, x_2) = \sum_{i=1}^{N} \mathbf{z}_i^T \, . \, \boldsymbol{\psi}_i(x_1, x_2) \tag{23}$$

where $\mathbf{z}_i = \boldsymbol{\partial} z(P_i)$. Practically, one gives the vector $\mathbf{z}_i$ at each meshpoint of the domain $\bar{D}$, which entirely define the surface. On the other hand, for our variational problem, we restrict the space of the three unknown functions u_1, u_2, u_3 to be a space of finite dimension, of the form

$$u_i(x_1, x_2) = \sum_{j=1}^{N} \boldsymbol{\partial}^T u_i(P_j) \, . \, \boldsymbol{\psi}_j(x_1, x_2), \qquad (i = 1, 2, 3) \tag{24}$$

With such a choice of admissible functions one can represent exactly the rigid-body motions of the surface. Indeed, a linearized rigid-body displacement may be written $\tilde{\mathbf{u}} = \mathbf{u}_0 + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0)$, where $\mathbf{r}^T = [x_1, x_2, z(x_1, x_2)]$; the components of $\tilde{\mathbf{u}}$ are therefore of the form $\tilde{u}_i = a_0 + a_1 x_1 + a_2 x_2 + a_3 z$ and, for such a function, one has the equality

$$\sum_{j=1}^{N} \boldsymbol{\partial}^T \tilde{u}_i(P_j) \, . \, \boldsymbol{\psi}_j(x_1, x_2) = \tilde{u}_i(x_1, x_2)$$

The proof is immediate: set $\tilde{u}_i = v + a_3 z$, v is a polynomial of the first order in x_1, x_2 for which we have the relation (22) and, from its definition, $z(x_1, x_2)$ has the form (23).

Besides, we show in the second section that the strain energy vanishes if, and only if, the middle surface of the shell undergoes a rigid-body displacement. It follows that the formulas (23) and (24) represent the rigid-body motions of the shell exactly.

It is convenient for the following to restrict the functions ψ_i on an element. Let Δ be the triangle of $\bar{D}$, admitting the vertex P_r, P_s, P_t and let us note $v(x_1, x_2)$ a function defined on Δ by the formulas (23) or (24). In order to lighten the writing, we introduce the vector $\mathbf{v}$, relative to the element Δ, defined as

$$\mathbf{v}^T = [\partial^T v(P_r), \partial^T v(P_s), \partial^T v(P_t)]$$

If we denote by $\boldsymbol{\phi}(x_1, x_2)$ the $3n$ functions ψ_r, ψ_s, ψ_t defined in Δ only, then $v(x_1, x_2)$ takes the form

$$v(x_1, x_2) = \mathbf{v}^T \boldsymbol{\phi}(x_1, x_2) \tag{25}$$

Now, let us consider a linear mapping such that the triangle P_r, P_s, P_t of the plane (x_1, x_2) is mapped on the unit triangle of the plane (ξ_1, ξ_2) whose vertices are (0, 0), (1, 0), (0, 1) (see Fig. 2), defined by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} x_{1r} \\ x_{2r} \end{pmatrix},$$

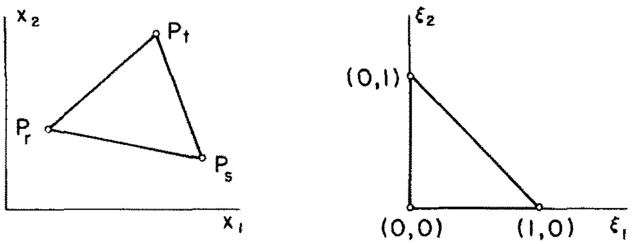

FIG. 2.

where (26)

$$C = \begin{pmatrix} x_{1s} - x_{1r} & x_{1t} - x_{1r} \\ x_{2s} - x_{2r} & x_{2t} - x_{2r} \end{pmatrix}$$

(x_{1r}, x_{2r}) being the coordinates of the node P_r. We have pointed out in [21] that the functions $\boldsymbol{\phi}$ relative to the triangle Δ, can be put under the form

$$\boldsymbol{\phi}(x_1, x_2) = \mathbf{T}\tilde{\boldsymbol{\phi}}(\xi_1, \xi_2) \tag{27}$$

the set (x_1, x_2) and (ξ_1, ξ_2) being linked by the relations (26). The matrix $\mathbf{T}$ characterizes the geometry of the triangle and $\tilde{\boldsymbol{\phi}}(\xi_1, \xi_2)$ are some functions defined on the unit triangle of the plane (ξ_1, ξ_2). The functions of the form (25) can therefore be written as

$$v(x_1, x_2) = \mathbf{v}^T \mathbf{T}\tilde{\boldsymbol{\phi}}(\xi_1, \xi_2) \tag{28}$$

The functions $\tilde{\boldsymbol{\phi}}(\xi_1, \xi_2)$ are called basic functions of the plane (ξ_1, ξ_2). In the following, we shall use the three types of basic functions given in [21] and summarized in Table 3. For our variational problem, they define the sub-spaces of admissible functions of dimension $18N$, $18N$ and $9N$ respectively.

TABLE 3

Type	n	Parameters associated at each node (vector $\mathbf{v}$)	Dimension of the matrix $\mathbf{T}$	Functions $\tilde{\boldsymbol{\phi}}$	Class
$T1$	6	$v, v_{x_1}, v_{x_2}, v_{x_1x_1}, v_{x_2x_2}, v_{x_1x_2}$	18×30	Rational functions	C^2
$T2$	6	$v, v_{x_1}, v_{x_2}, v_{x_1x_1}, v_{x_2x_2}, v_{x_1x_2}$	18×21	Polynomials of the 5th degree	C^1
$T3$	3	v, v_{x_1}, v_{x_2}	9×12	Rational functions	C^1

Remark

Koiter's model reviewed in the second section only involves the derivatives of second order of the function z. Of course, basic functions of class C^1 are sufficiently regular in this case. However, in some other models (for example, that given by Green and Zerna [20]), the expression of strain energy in Cartesian coordinates, make use of the derivatives of z of third order and there, it is necessary to use basic functions of class C^2.

5. DERIVATION OF THE STIFFNESS MATRIX OF AN ELEMENT

We now propose to calculate the contribution of an element to the strain energy (20), restricting the admissible functions to those of the form (24).

The contribution of the element Δ is

$$\Delta U_1 = \int\int_{\Delta} \sum_{i,j=1}^{3} \partial_x^T u_i \mathbf{R}_{ij} \partial_x u_j \sqrt{a}\, dx_1\, dx_2 \tag{29}$$

with $\mathbf{R}_{ij} = C_m \mathbf{A}_i^T \mathbf{K} \mathbf{A}_j + C_f \mathbf{B}_i^T \mathbf{K} \mathbf{B}_j$. Let us effect the change of variable (26) in the integral (29). One has the formulas of derivation

$$\partial_x v = \mathbf{S} \partial_\xi v \tag{30}$$

where $\mathbf{S}$ is a 6×6 matrix depending on the geometry of the triangle Δ, given at Table 4, with the notations

$$t_1 = (x_{2t} - x_{2r})/J, \qquad t_2 = -(x_{1t} - x_{1r})/J,$$
$$t_3 = -(x_{2s} - x_{2r})/J, \qquad t_4 = (x_{1s} - x_{1r})/J;$$
$$J = (x_{1s} - x_{1r})(x_{2t} - x_{2r}) - (x_{1t} - x_{1r})(x_{2s} - x_{2r})$$

TABLE 4

$\mathbf{S} =$

1					
	t_1	t_3			
	t_2	t_4			
			t_1^2	t_3^2	$2t_1t_3$
			t_2^2	t_4^2	$2t_2t_4$
			t_1t_2	t_3t_4	$t_1t_4 + t_2t_3$

being the Jacobian of the transformation. Substitution of (30) into (29) gives

$$\Delta U_1 = \int\int \sum_{i,j=1}^{3} \partial_\xi^T u_i \bar{\mathbf{R}}_{ij} \partial_\xi u_j \sqrt{a} |J| \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \tag{31}$$

where $\bar{\mathbf{R}}_{ij} = C_m \bar{\mathbf{A}}_i^T \mathbf{K} \bar{\mathbf{A}}_j + C_f \bar{\mathbf{B}}_i^T \mathbf{K} \bar{\mathbf{B}}_j$, with $\bar{\mathbf{A}}_i = \mathbf{S}\mathbf{A}_i$ and $\bar{\mathbf{B}}_i = \mathbf{S}\mathbf{B}_i$. Let us now introduce the admissible functions (24) in this integral. Writing henceforth $\boldsymbol{\phi}$, rather than $\tilde{\boldsymbol{\phi}}$, the basic function of the plan (ξ_1, ξ_2), we find

$$\Delta U_1 = |J| \sum_{i,j=1}^{3} \mathbf{u}_i^T \mathbf{T} \left\{ \sum_{k,l=1}^{6} \int\int \partial_k \boldsymbol{\phi} \partial_l \boldsymbol{\phi}^T \bar{R}_{ijkl} \sqrt{a} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \right\} \mathbf{T}^T \mathbf{u}_j.$$

The elements $\bar{R}_{ijkl}$ of the matrix $\bar{\mathbf{R}}_{ij}$ depend in a nonlinear way, on the geometry of the surface. In order to enable us to effect the numerical integration once and for all, we interpolate this functions as follows

$$\bar{R}_{ijkl}(\xi)\sqrt{[a(\xi)]} = \sum_{p=1}^{m} \bar{R}_{ijkl}(I_p)\sqrt{[a(I_p)]}\theta_p(\xi) \tag{32}$$

where θ_p are Lagrangian polynomials of interpolation, relative to the points I_p of the unit triangle and ξ stands for ξ_1, ξ_2. The contribution of the element to the strain energy becomes then

$$\Delta U_1 = |J| \sum_{i,j=1}^{3} \mathbf{u}_i^T \mathbf{T} \left\{ \sum_{k,l=1}^{6} \sum_{p=1}^{m} \bar{R}_{ijkl}(I_p)\sqrt{[a(I_p)]}\mathbf{G}_{klp} \right\} \mathbf{T}^T \mathbf{u}_j \tag{33}$$

where $\mathbf{G}_{klp}$ are the matrices

$$\mathbf{G}_{klp} = \int\int \partial_k \boldsymbol{\phi}(\xi) \partial_l \boldsymbol{\phi}^T(\xi) \theta_p(\xi) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \tag{34}$$

which depend on the choice of the basic functions and the Lagrangian polynomials. For a given set of such functions, these integrals may be computed once and for all. Finally, the stiffness matrix of the element is defined by the relation

$$\Delta U_1 = |J| \sum_{i,j=1}^{3} \mathbf{u}_i^T \mathbf{T} \mathbf{Q}_{ij} \mathbf{T}^T \mathbf{u}_j,$$

with

$$\mathbf{Q}_{ij} = \sum_{k,l=1}^{6} \sum_{p=1}^{m} \bar{R}_{ijkl}(I_p)\sqrt{[a(I_p)]}\mathbf{G}_{klp} \tag{35}$$

6. PRACTICAL ASPECTS OF COMPUTATION

Practically an element of shell is defined by its coordinates in the plane (x_1, x_2), its vector $\mathbf{z}$ and the thickness on the vertices.

We make use of Lagrangian polynomials of the third order, relative to the 10 points of interpolation shown in Fig. 3. At each such point, we have to form the matrices $\mathbf{A}_i$, $\mathbf{B}_i$ $(i = 1, 2, 3)$ and $\mathbf{K}$ entirely defined by the geometry of the element. For this purpose, we compute the values of the function z and its derivatives at these points from the basic functions and the vector $\mathbf{z}$. We get then the matrices $\bar{\mathbf{R}}_{ij}$, from which we draw the stiffness matrix.

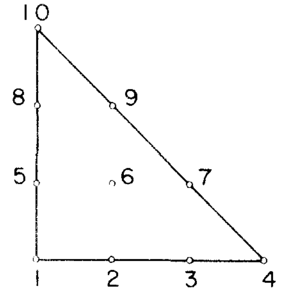

FIG. 3.

To estimate numerically the integrals $\mathbf{G}_{klp}$, we use the 7-points formula given in [23]. We found a satisfying accuracy on using these formulas on 64 sub-triangles by dividing each side of the unit triangle in 8 equal parts. Those coefficients are computed once and for all and kept on a tape.

The interpolation of functions $\bar{R}_{ijkl}$ by means of the polynomials θ_p, yields that rigid-body motions cannot be represented exactly. However, some numerical experiments show that we get a very good approximation with the 10 points mentioned below, as soon as the mesh is rather fine (see [24]).

This element of shell has been introduced in a general purpose program, developed for the IBM 7040 computer of the EPFL (see [22]). This program deals with the formation of the master stiffness matrix and right-hand side of the structure, taking account of the boundary conditions; with the solving of the linear equations and the computation of stresses. One can introduce any linear conditions between the degrees of freedom of the structure and assemble elements of various kinds such as beam, plate, shell, etc. For the elements of shell, the program computes the physical components of in-plane and bending stresses at the corners and in the middle of the element. With the $T3$ basic functions, the stresses are not continued at the nodes; in that case, one computes the average stresses at a node from the elements admitting this node for vertex.

7. NUMERICAL EXAMPLE

We consider the shell shown in Fig. 4; it is defined by the equation $z = 5 - x_1^2/20$, $-10 \leq x_1 \leq 10$, $-10 \leq x_2 \leq 10$; its uniform thickness is $h = 0{\cdot}2$, and the elastic coefficients are $E = 2 \times 10^7$, $\nu = 0{\cdot}15$.

This cylinder is supported along the edges $x_2 = \pm 10$, in such a way that we have $u_1 = u_3 = 0$; it is free along the edges $x_1 = \pm 10$. We propose to settle the field of displacement and the state of stress under a uniform pressure $p_3 = -1$.

We have computed the quarter of the shell, with the three meshes shown in Fig. 5 and the three types of basic functions $T1$, $T2$ and $T3$. Some characteristic numerical results are given in Tables 5 and 6. From these results, we can draw the following conclusions:

1. With a coarse mesh, the elements $T1$ generally give a better approximation than the elements $T2$. The results are almost the same when the mesh becomes fine.

2. The elements $T3$ which have only 9 degrees of freedom at each node, instead of 18 for $T1$ and $T2$, lead to worse numerical results for a given time of computation.

3. If the mathematical model only requires basic functions of class C^1, the elements $T2$ seems to be the best one.

TABLE 5

Point	Elements / Displacement	Mesh 1 [Fig. 5(a)]			Mesh 2 [Fig. 5(b)]			Mesh 3 [Fig. 5(c)]		
		*T*1	*T*2	*T*3	*T*1	*T*2	*T*3	*T*1	*T*2	*T*3
A	$u_1 \times 10^2$	0·842	0·731	0·476	0·772	0·760	0·722	0·7663	0·7662	0·7613
	$u_3 \times 10^2$	−1·117	−0·999	−0·658	−1·039	−1·024	−0·975	−1·0315	−1·0315	−1·0251
B	$u_1 \times 10^2$	0·110	0·130	0·087	0·114	0·115	0·112	0·1144	0·1146	0·1143
	$u_3 \times 10^2$	−0·165	−0·203	−0·143	−0·175	−0·177	−0·174	−0·1759	−0·1761	−0·1760
C	$u_3 \times 10^2$	0·155	0·179	0·109	0·159	0·160	0·155	0·1594	0·1596	0·1589

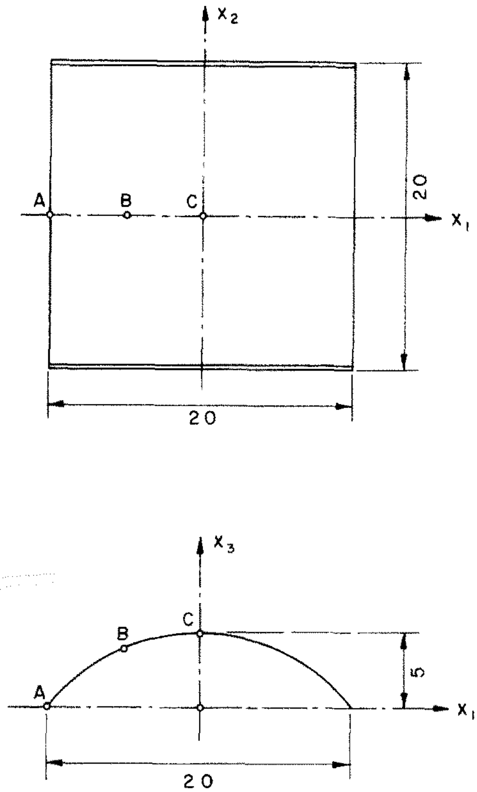

FIG. 4.

TABLE 6

Point	Elements / Stress	Mesh 1 [Fig. 5(a)] T2	Mesh 1 [Fig. 5(a)] T3	Mesh 2 [Fig. 5(b)] T2	Mesh 2 [Fig. 5(b)] T3	Mesh 3 [Fig. 5(c)] T2	Mesh 3 [Fig. 5(c)] T3
A	$n_{(22)}$	280·0	125·7	272·2	228·9	272·4	261·6
	$m_{(22)}$	3·73	2·38	3·73	3·63	3·74	3·93
B	$n_{(22)}$	−69·2	−42·4	−69·2	−66·9	−69·8	−69·3
	$m_{(22)}$	0·78	0·97	0·25	0·56	0·21	0·30
C	$n_{(22)}$	55·1	64·9	48·1	57·9	48·0	50·6
	$m_{(22)}$	−1·03	−0·82	−1·08	−1·04	−1·06	−1·01

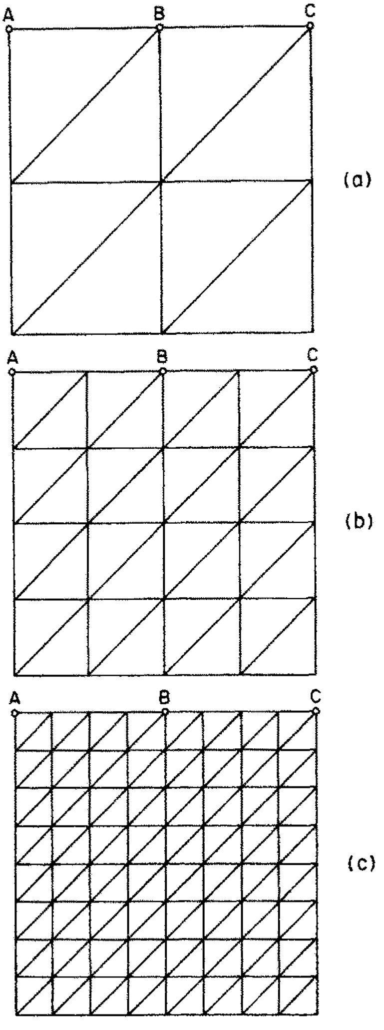

FIG. 5.

Acknowledgements—The authors are indebted to Professor J. Descloux, EPFL—Switzerland, for helpful discussions and to Professor W. Prager, Brown University, for his help in correcting the English translation of the manuscript.

REFERENCES

[1] O. C. ZIENKIEWICZ, *The Finite Element Method in Structural and Continuum Mechanics*. McGraw-Hill (1967).
[2] C. P. JOHNSON, A finite element approximation of the analysis of thin shells, Ph.D. Dissertation, University of California, Berkeley (1967).
[3] A. J. CARR, A refined finite element analysis of thin shell structures including dynamic loadings, Ph.D. Dissertation, University of California, Berkeley (1967).
[4] R. W. CLOUGH and C. P. JOHNSON, A finite approximation for the analysis of thin shells. *Int. J. Solids Structures* **4**, 43–60 (1968).
[5] O. C. ZIENKIEWICZ, C. PAREKH and I. P. KING, Arch dams analysed by a linear finite element shell solution program, *Proc. Symp. on Arch Dams*, pp. 19–22. Inst. of Civil Eng., London (1968).
[6] J. ERGATOUDIS, B. M. IRONS and O. C. ZIENKIEWICZ, Three dimensional stress analysis of arch dams by the finite element method, Reports AD/1935 and AD/1745, Inst. of Civil Eng., London (1966).

[7] J. Ergatoudis, B. M. Irons and O. C. Zienkiewicz, Three dimensional stress analysis of arch dams and their foundations, *Proc. Symp. on Arch Dams*, pp. 37–50. Inst. of Civil Eng., London (1968).

[8] S. Ahmad, B. M. Irons and O. C. Zienkiewicz, Curved thick shell and membrane element with particular reference to axi-symmetric problems, *Proc. 2nd Conf. on Matrix Methods in Struct. Mech.* Wright–Patterson Air Force Base, Ohio (1968).

[9] S. Ahmad, Curved finite elements in the analysis of solid, shell and plate structures, Ph.D. Dissertation, University of Wales, Swansea (1969).

[10] S. Ahmad, B. M. Irons and O. C. Zienkiewicz, Analysis of thick and thin shell structures by general curved elements with special reference to arch dams, Research Report C/R/99/69, University of Wales, Swansea (1969).

[11] G. E. Strickland and W. A. Loden, A doubly-curved triangular shell element, *Proc. 2nd Conf. on Matrix Methods in Struct. Mech.* Wright–Patterson Air Force Base, Ohio (1968).

[12] B. E. Greene, R. E. Jones and D. R. Stone, Dynamic analysis of shell using doubly curved finite elements, *Proc. 2nd Conf. on Matrix Methods in Struct. Mech.* Wright–Patterson Air Force Base, Ohio (1968).

[13] J. J. Connor and C. Brabbia, Stiffness matrix for shallow rectangular shell elements, *J. Engng. Mech. Div. Am. Soc. Civ. Engrs.* **93**, 43–65 (1967).

[14] S. Utku, Stiffness matrices for thin triangular element of nonzero Gaussian curvature. *AIAA Jnl* 5, 1659 1667 (1967).

[15] H. Fette, Gekrümmte finite Elemente zur Berechnung von Shalentragwerken, Shriftenreihe der Institut für Konstructiven Igenieurbau, der Technishen Universität Braunshweig, Helft 1, Düsseldorf (1969).

[16] Ph. Clement and J. Descloux, On the rigid displacement condition, submitted to *Int. J. Num. Meth. Engng.*

[17] J. J. Goël, Construction of basic functions for numerical utilisation of Ritz's method. *Num. Math.* **12**, 435–447 (1968).

[18] *The Theory of Thin Elastic Shells*, Proc. Symp. IUTAM, Delft (1959), Ed. by W. T. Koiter. North-Holland (1960).

[19] *Theory of Thin Shells*, Proc. Symp. IUTAM, Copenhagen (1967), Ed. by F. L. Niordson. Springer-Verlag (1969).

[20] A. E. Green and W. Zerna, *Theoretical Elasticity*. Oxford Press (1954).

[21] G. Dupuis and J. J. Goël, Finite elements with high degree of regularity. To appear *Int. J. Num. Meth. Engng.*

[22] C. Bossoney, G. Dupuis and J. J. Goël, CASTEL a general purpose program for structural analysis, Technical Report, Ecole Polytechnique Fédérale-Lausanne, Department of Mathematics, Lausanne (1969).

[23] P. C. Hammer, O. J. Marlove and A. H. Strond, Numerical integration over simplex and cones. MTAC **X**, 130–136 (1956).

[24] G. Dupuis and J. J. Goël, Some numerical results in finite element analysis of thin elastic shells, Technical Report, Ecole Polytechnique Fédérale-Lausanne, Department of Mathematics, Lausanne (1969).

(*Received 7 January* 1970)

Résumé—Cet article concerne un élément fini de coque, triangulaire, courbe qui permet de représenter exactement les déplacements rigides et assure la convergence en énergie. La matrice de rigidité est obtenue de telle manière que les calculs sont valables pour tous les modèles mathématiques construits à partir de l'hypothèse de Kirchhoff. Un exemple numérique indique la qualité des résultats obtenus avec 9 ou 18 degrés de liberté associés à chaque noeuds du réseau et des fonctions de base de classe C^1 ou C^2.

Абстракт—Работа касается расчета криволинейного трехугольного конечного элемента оболочки, который определяет точно движение жесткого тела и обеспечивает сходимость в выражении для энергии. Общим путем выводится матрица коэффициентов жесткости, что является важным для всех математических моделей в области применения предположения Кирхгоффа. Дается численный пример с целью указания качественной стороны результатов, которые можно получить для 9 или 18 степеней свободы в каждой точке сетки и основных функций класса C^1 или C^2.